Lesson Thirty Six

Sequences

Aims

The aims of this lesson are to help you to:

- recognise arithmetic and geometric sequences
- generate these sequences from 'term to term' or 'position to term' rules
- find missing terms in these sequences
- recognise quadratic and other sequences

Why am I studying this?

Arithmetic and geometric sequences appear in areas such as finance, when calculating how much interest is to be paid.

Oxford Home Schooling

Introduction

The following are examples of sequences.

(1) $3, 5, 7, 9, 11, \ldots$

(2) $9, 5, 1, -3, -7, \ldots$

(3) $2, 4, 8, 16, 32, \ldots$

(4) $8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

(5) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

(6) $1, 4, 9, 16, 25, \ldots$

(7) $-1, 2, 7, 14, 23, \ldots$

In each case there is a *pattern* behind the numbers. Once the pattern has been found, further terms in the sequence can be determined.

In (1), we are just adding 2 each time. This is an example of an **arithmetic** sequence with **common difference** 2.

An arithmetic sequence can have a negative common difference, as in (2), where we are subtracting 4 each time.

In (3) we are multiplying by 2 each time. This is an example of a **geometric** sequence with **common ratio** 2. In (4) the common ratio is $\frac{1}{2}$.

These are the main types of sequence that we will be looking at in this lesson. However, other types are possible. In some cases the pattern is easier to spot than others.

In (5), for example, we can say that the nth term is $\frac{1}{n}$ (the **reciprocal** of n), whilst in (6) it is n^2. You will recall that n stands for *any* number.

In (7), the nth term is $n^2 - 2$.

Sequences (6) and (7) are described as **quadratic** (as they involve an n^2 term).

When listed, each term in a sequence is separated by a comma and the three dots at the end (i.e. the 'ellipsis') indicate that further terms in the sequence *could* have been written down – generally an infinite number of further terms.

Activity 1	Describe each of the following sequences, as fully as possible.
	1. $8, 3, -2, -7, -12, \ldots$ 2. $48, 12, 3, \frac{3}{4}, \frac{3}{16}, \ldots$ 3. $-2, 6, -18, 54, -162, \ldots$ 4. $2, 9, 28, 65, 126, \ldots$

Arithmetic Sequences

If we know the first term of an arithmetic sequence and the common difference, then we can create as many further terms as we wish, simply by adding on the common difference each time. The common difference is sometimes referred to as the '**term to term**' rule.

The n^{th} term of an arithmetic sequence can be obtained by adding $n - 1$ lots of the common difference to the first term.

For example, if the first term is 3 and the common difference is 2 (as in Example (1) above), then the 5th term (with $n = 5$) equals $3 + [(n - 1) \times 2] = 3 + (4 \times 2) = 11$

This is sometimes called the '**position to term**' rule: the position, n determines the value of the term.

Notice that it also works for the first term, when $n = 1$,

as $3 + (1 - 1) \times 2 = 3 + 0 = 3$

If we call the first term a and the common difference d, then this formula for the n^{th} term can be written as:

$a + (n - 1)d$

Example 1

If the first term of an arithmetic sequence is 12 and the common difference is 3, then the formula for the n^{th} term is:

$12 + [(n - 1) \times 3]$

This can be simplified to $12 + 3n - 3 = 9 + 3n$ or $3n + 9$.

Example 2

If the n^{th} term of an arithmetic sequence is $4n + 3$, then we can write $a + (n - 1)d = 4n + 3$,

which means that d has to equal 4 (in order to obtain $4n$ on the right-hand side), and $a - d = 3$, so that $a = 7$

Also, by putting $n = 1$ in the formula $4n + 3$, we get $a = 7$.

Check: When $n = 4$ (say), $4n + 3 = 16 + 3 = 19$,

and $7 + (4 - 1) \times 4 = 7 + 12 = 19$

Activity 2	Obtain a formula for the n^{th} term of each of the following arithmetic sequences (giving the formula in its simplest form).

	1. $5, 9, 13, 17, 21, \ldots$ 2. $16, 14, 12, 10, 8, \ldots$ 3. $2, 7, 12, 17, 22, \ldots$

Activity 3	For each of the following arithmetic sequences (given by the formula for the n^{th} term), find the common difference and first term.
	1. $9n + 5$ 2. $3n - 2$ 3. $n + 1$ 4. $6 - 2n$

Geometric Sequences

Example 3

Find the formula for the n^{th} term of the following sequence: $3, 6, 12, 24, 48, \ldots$

This is a geometric sequence with common ratio 2. The n^{th} term is obtained by multiplying the first term by 2, $n - 1$ times:

$3 \times 2^{n-1}$

Notice that, when $n = 1$, we have $3 \times 2^0 = 3 \times 1 = 3$, as expected.

Activity 4	Now try the following.
	1. Write down the formula for the nth term of the following sequence: $1, 3, 9, 27, 81, \ldots$ 2. Write down the first 5 terms of the sequence for which the nth term is $64 \times \left(\frac{1}{2}\right)^{n-1}$ 3. Find the 3rd and 4th terms of the following geometric sequence: 500, 100, _ , _ , 0.8 , ...

Other Sequences

Quadratic sequences can be identified by finding the '2nd differences', as shown in the following example.

Example 4

Consider the sequence with n^{th} term equal to $4n^2 + 3$. The first 5 terms are: $7, 19, 39, 67, 103$

The 1st differences are:

$19 - 7 = 12$

$39 - 19 = 20$

$67 - 39 = 28$

$103 - 67 = 36$

The 2nd differences are:

$20 - 12 = 8$

$28 - 10 = 8$

$36 - 28 = 8$

If you experiment with similar sequences yourself you will find that they always include the expression kn^2, where k is half the 2nd difference, whenever the 2nd difference is a constant number.

Example 5 (advanced)

To find the n^{th} term of the sequence: $3, 6, 13, 24, 39$

The 1st differences are: $3, 7, 11, 15$

The 2nd differences are: $4, 4, 4$

So we know that the formula for the n^{th} term includes the expression $2n^2$.

If we subtract $2n^2$ from each term of the sequence, we get:

$3 - 2(1)^2 = 1$

$6 - 2(2)^2 = -2$

$13 - 2(3)^2 = -5$

$24 - 2(4)^2 = -8$

$39 - 2(5)^2 = -11$

You can check that this is the arithmetic sequence $4 - 3n$.

Hence the original sequence is $2n^2 - 3n + 4$

Activity 5	Find the n^{th} term of the sequence of 'triangular' (or 'triangle') numbers: $1, 3, 6, 10, 15, \ldots$, by finding the 2nd differences. [This sequence gets its name from the fact that equilateral triangles can be formed from 1, 3, 6, 10, 15, ... dots, by adding a new row underneath the previous triangle. The n^{th} term also equals $1 + 2 + 3 + \cdots + n$]

Activity 6 (advanced)	Find a similar procedure for dealing with sequences that include a term involving n^3.

Example 6

Another well-known sequence is the **Fibonacci** sequence:

$1, 1, 2, 3, 5, 8, 13, 21, \ldots$ (the rule is this: after the first two terms, each term is obtained by adding the previous two)

Activity 7	Investigate the ratio of the n^{th} term of the Fibonacci sequence to the $(n-1)^{\text{th}}$ term (go up to $n = 20$) [this is most easily done using an Excel spreadsheet]

Suggested Answers to Activities

Activity One

1. Arithmetic, with common difference -5
2. Geometric with common ratio $\frac{1}{4}$
3. Geometric with common ratio -3
4. The sequence with n^{th} term equal to n^3+1

Activity Two

1. $\boldsymbol{d=4}$; $\boldsymbol{n}^{\text{th}}$ term is $\boldsymbol{5+(n-1)\times 4=5+4n-4=4n+1}$

   Check: When $\boldsymbol{n=5, 4n+1=21}$

2. $\boldsymbol{d=-2}$; $\boldsymbol{n}^{\text{th}}$ term is $\boldsymbol{16+(n-1)\times(-2)=16-2n+2=18-2n}$

   Check: When $\boldsymbol{n=5, 18-2n=8}$

3. $d=5$; n^{th} term is $2+(n-1)\times 5=2+5n-5=5n-3$

   Check: When $n=5, 5n-3=22$

Activity Three

1. $9n+5=a+(n-1)d$

   So $d=9$ and $5=a-d$; i.e. $a=5+9=14$

   Check: When $n=3$ (say), $9n+5=27+5=32$

   and $14+(3-1)\times 9=14+18=32$

2. $3n-2=a+(n-1)d$

   So $d=3$ and $-2=a-d$; i.e. $a=-2+3=1$

   Check: When $n=4$ (say), $3n-2=12-2=10$

   and $1+(4-1)\times 3=1+9=10$

3. $n+1=a+(n-1)d$

   So $d=1$ and $1=a-d$; i.e. $a=1+1=2$

   Check: When $n=5$ (say), $n+1=5+1=6$

   and $2+(5-1)\times 1=2+4=6$

4. $6-2n=a+(n-1)d$

   So $d=-2$ and $6=a-d$; i.e. $a=6-2=4$

   Check: When $n=6$ (say), $6-2n=6-12=-6$

   and $4+(6-1)\times(-2)=4-10=-6$

Activity Four

1. 3^{n-1}
2. $64, 32, 16, 8, 4$
3. 500, 100, 20, 4, 0.8, ...

Activity Five

The 1st differences are: $2, 3, 4, 5$

The 2nd differences are: $1, 1, 1$

So the nth term includes the expression $\frac{1}{2}n^2$

Subtracting $\frac{1}{2}n^2$ from each term gives:

$1 - \frac{1}{2}(1)^2 = \frac{1}{2}$

$3 - \frac{1}{2}(2)^2 = 1$

$6 - \frac{1}{2}(3)^2 = \frac{3}{2}$

$10 - \frac{1}{2}(4)^2 = 2$

$15 - \frac{1}{2}(5)^2 = \frac{5}{2}$

which leaves the arithmetic sequence $\frac{1}{2}n$,

so that the required formula is $\frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n + 1)$

Activity Six

Divide the 3rd difference by 6, to find the multiplier of n^3. Then subtract the term involving n^3, to leave a quadratic sequence, and carry on as before.

Activity Seven

The ratio gets closer and closer to the number 1.618033989 ... (which can be shown to equal $\frac{1+\sqrt{5}}{2}$)